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Time-dependent motion of a two-dimensional floating elastic plate

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Abstract

Three methods are presented to determine the motion of a two-dimensional finite elastic plate floating on the water surface, which is released from rest and allowed to evolve freely. The first method is based on a generalized eigenfunction expansion and it is valid for all water depths. The second method is based on an integral equation derived from the Fourier transform, and it is valid for all water depths, although computations are made only for water of infinite depth. These two methods are both based on the frequency-domain solution—however no obvious connection exists between the two methods. The third method is valid only for shallow water, and it expresses the solution as the sum over decaying modes. We present a new derivation of the integral equation for a floating plate based on the Fourier transform of the equations of motion in the time domain. The solution obtained by each method is compared in the appropriate regime, and excellent agreement is found, thereby providing benchmark solutions. We also investigate the regime of validity of the infinite and shallow-depth solutions, and show that both give good results for a quite wide range of depths.

Keywords: Floating elastic plates; VLFS; Time-domain; Linear water waves

1. Introduction

The floating elastic plate is a very natural problem to consider when attempting to understand hydroelastic motions. As well as being a simple model for practical structures such as a very large floating structure (VLFS), the floating elastic plate serves as a simple model for more complicated and realistic geometries. For this reason the floating elastic plate is amongst the best-studied problems in hydroelasticity, and it has been used to model floating breakwaters (Stoker, 1957), ice floes (Squire et al., 1995; Squire, 2007) and very large floating structures (Kashiwagi, 2000a; Watanabe et al., 2004).

The single-frequency (time harmonic) response for a floating elastic plate in shallow water was presented in Stoker (1957) and the solution for finite depth was given in Meylan and Squire (1994) and Newman (1994) for two dimensions. A detailed discussion of the three-dimensional problem can be found in Squire (2007). The time-dependent problem is much more challenging, especially for the case of a plate released from rest which is allowed to evolve freely. A solution

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for the case of shallow water has been presented in Meylan (2002) using a generalized eigenfunction expansion and by Lax–Phillips scattering theory, the latter of which provides the solution as a sum over modes which oscillate and decay. A solution for shallow water was also given in Sturova (2002), by solving a differential equation governing the temporal evolution of the free modes of the plate. Kashiwagi (2000b, 2004) used the integral equation method to carry out a numerical simulation of the transient responses of a rectangular VLFS floating on deep water, during landing and take-off by an aeroplane. Recently Sturova (2006) has developed this integral equation method for the time-dependent problem, for a floating elastic plate on water of infinite depth. But no derivation of the integral equations for a floating elastic plate was given in those papers, and we present such a derivation here. Recently Hazard and Meylan (2007) have extended the generalized eigenfunction method to water of finite depth. A solution to the problem by a finite-element method is given in Qui (2007), and the solution for a circular plate on shallow water is given in Sturova (2003). The two-dimensional unsteady problem of the hydroelastic behaviour of a plate floating on water of infinite depth was solved in Korobkin (2000), but gravity effects were neglected in determining the maximum of both the plate deflection and bending stresses in the plate.

In this paper, we present the time-dependent solution to the floating elastic plate released from rest by three methods. The first method is based on a generalized eigenfunction expansion and it is valid for water of finite depth. The second method is based on an integral equation which was given by Ogilvie (1964) for a rigid body, although his derivation is different from the derivation we present here which based on the Fourier transform. This method is sometime called ‘‘the memory effect equation’’. The second method is valid for finite depth, but the numerical solution is presented only for infinite depth. The third solution method is for shallow water, and it expresses the solution as a sum of modes. The first two solutions require the solution in the frequency domain. We present numerical solutions which show good agreement between the three numerical methods (in the appropriate depths), thus providing benchmark solutions. We also investigate the range of depths for which both the infinite-depth and shallow-water assumptions are valid.

2. Statement of the problem and mathematical formulations

2.1. Governing equations

The plate is infinite in the y direction, so that only the x and z directions are considered. The x direction is horizontal, the positive z axis points vertically up, and the plate covers the region $-b \leqslant x \leqslant b$. The water is of uniform depth h. The amplitudes are assumed small enough that the linear theory is appropriate, and the plate is sufficiently thin that the shallow-draft approximation may be made (Watanabe et al., 2004). The mathematical description of this problem follows from Stoker (1957). The velocity potential Φ satisfies

$$\Delta\Phi = 0, \quad -h < z < 0, \tag{1}$$

$$\partial_z\Phi = 0, \quad z = -h. \tag{2}$$

The kinematic condition is

$$\partial_t\xi = \partial_z\Phi, \quad z = 0, \tag{3}$$

where ξ is the displacement of the water surface or the plate (from the shallow-draft approximation). The dynamic condition, obtained by matching the pressure at the free surface, is

$$-\rho g\xi - \rho\partial_t\Phi = 0, \quad x \notin (-b, b), \quad z = 0, \tag{4}$$

$$-\rho g\xi - \rho\partial_t\Phi = D\partial_x^4\xi + \rho' d\partial_t^2\xi, \quad x \in (-b, b), \quad z = 0, \tag{5}$$

where D is the bending rigidity of the plate per unit length, ρ is the density of water, ρ' is the density of the plate, d is the plate thickness and g is the acceleration due to gravity. At the ends of the plate the free-edge boundary conditions

$$\lim_{x\downarrow -b}\partial_x^2\xi = \lim_{x\uparrow b}\partial_x^2\xi = \lim_{x\downarrow -b}\partial_x^3\xi = \lim_{x\uparrow b}\partial_x^3\xi = 0 \tag{6}$$

are applied.

Nondimensional variables are now introduced, using a length parameter L for the space variables, and $\sqrt{L/g}$ for the time variable. We leave the choice of the length parameter arbitrary, since there are two natural length parameters, the water depth and the characteristic length $(D/\rho g)^{1/4}$. It also means that we can present results in our nondimensional variables, in which the plate properties are kept constant and the water depth is varied. Hence the nondimensional surface displacement and velocity potential satisfy the following coupled equations, where the overbar denotes nondimensional variables,

$$\bar{\Delta}\bar{\Phi} = 0, \quad -\bar{h} < \bar{z} < 0, \quad \partial_{\bar{z}}\bar{\Phi} = 0, \quad \bar{z} = -\bar{h}, \tag{7a,b}$$

$$\partial_{\bar{t}}\bar{\xi} = \partial_{\bar{z}}\bar{\Phi}, \quad \bar{z} = 0, \tag{7c}$$

$$-\bar{\xi} - \partial_{\bar{t}}\bar{\Phi} = 0, \quad \bar{x} \notin (-\bar{b}, \bar{b}), \quad \bar{z} = 0, \quad -\bar{\xi} - \partial_{\bar{t}}\bar{\Phi} = \beta\partial_{\bar{x}}^4\bar{\xi} + \gamma\partial_{\bar{t}}^2\bar{\xi}, \quad \bar{x} \in (-\bar{b}, \bar{b}), \quad \bar{z} = 0, \tag{7d,e}$$

plus the free-edge boundary conditions

$$\lim_{\bar{x}\downarrow-\bar{b}} \partial_{\bar{x}}^2\bar{\xi} = \lim_{\bar{x}\uparrow\bar{b}} \partial_{\bar{x}}^2\bar{\xi} = \lim_{\bar{x}\downarrow-\bar{b}} \partial_{\bar{x}}^3\bar{\xi} = \lim_{\bar{x}\uparrow\bar{b}} \partial_{\bar{x}}^3\bar{\xi} = 0, \tag{8}$$

where $\beta = D/(\rho g L^4)$ and $\gamma = \rho' d/(\rho L)$. For clarity the overbar is dropped from now on. Eqs. (7a)–(7e) are subject to the following initial conditions:

$$\xi(x,0) = \xi_0(x) \quad \text{and} \quad \partial_t\xi(x,0) = 0, \; x \in (-b,b), \tag{9}$$

as well as the condition that the initial fluid potential is at rest, i.e. $\Phi|_{t=0} = 0$. The problem of more general initial conditions is discussed in Hazard and Meylan (2007).

2.2. Expansion in modes

We expand the plate motion in the free (dry) modes which satisfy

$$\partial_x^4 w_n = \lambda_n^4 w_n, \tag{10}$$

and the free-edge conditions

$$\lim_{x\downarrow-b} \partial_x^2 w_n = \lim_{x\uparrow b} \partial_x^2 w_n = \lim_{x\downarrow-b} \partial_x^3 w_n = \lim_{x\uparrow b} \partial_x^3 w_n = 0, \tag{11}$$

where the eigenvalues λ_n ($n \geqslant 2$) denote the positive real roots of the equation

$$\tan(\lambda_n b) + (-1)^n \tanh(\lambda_n b) = 0, \tag{12}$$

and $\lambda_0 = \lambda_1 = 0$. The modes are given by

$$w_0 = \frac{1}{\sqrt{2b}}, \tag{13}$$

$$w_1 = x\sqrt{\frac{3}{2b^3}}, \tag{14}$$

$$w_{2n} = \frac{1}{\sqrt{2b}}\left(\frac{\cos(\lambda_{2n}x)}{\cos(\lambda_{2n}b)} + \frac{\cosh(\lambda_{2n}x)}{\cosh(\lambda_{2n}b)}\right), \tag{15}$$

and

$$w_{2n+1} = \frac{1}{\sqrt{2b}}\left(\frac{\sin(\lambda_{2n+1}x)}{\sin(\lambda_{2n+1}b)} + \frac{\sinh(\lambda_{2n+1}x)}{\sinh(\lambda_{2n+1}b)}\right). \tag{16}$$

The modes are normalized, i.e.

$$\int_{-b}^{b} w_m(x)w_n(x)\,\mathrm{d}x = \delta_{mn}, \tag{17}$$

where δ_{mn} is the Kronecker symbol. More details can be found in Sturova (2002). We expand the plate displacement in the modes as

$$\xi(x,t) = \sum_{n=0}^{\infty} A_n(t)w_n(x), \quad x \in (-b,b), \tag{18}$$

and substitute this expansion into (7a)–(7e), to obtain

$$\Delta\Phi = 0, \quad -h < z < 0, \quad \partial_z\Phi = 0, \quad z = -h, \tag{19a,b}$$

$$-\partial_t^2\Phi = \partial_z\Phi, \quad x \notin (-b,b), \quad z=0, \tag{19c}$$

$$\sum_{n=0}^{\infty}\partial_t\Lambda_n w_n = \partial_z\Phi, \quad x\in(-b,b), \quad z=0, \tag{19d}$$

$$\sum_{n=0}^{\infty}\Lambda_n(1+\beta\lambda_n^4)w_n + \gamma\sum_{n=0}^{\infty}\partial_t^2\Lambda_n w_n = -\partial_t\Phi, \quad x\in(-b,b), \quad z=0. \tag{19e}$$

The initial conditions are

$$\xi_0 = \sum_{n=0}^{\infty}\Lambda_n(0)w_n, \quad x\in(-b,b), \tag{20}$$

so that

$$\Lambda_n(0) = \int_{-b}^{b}\xi_0(x)w_n(x)\,dx \tag{21}$$

as well as the condition that $\partial_t\Lambda_n(t)|_{t=0}=0$ and that the fluid is initially at rest.

3. Single-frequency solution

Both methods we consider here rely on solving the single-frequency solution. For the floating plate this can be solved by a number of methods, including a Green's function method (Meylan and Squire, 1994; Newman, 1994) and an eigenfunction expansion (Hazard and Meylan, 2007; Kohout et al., 2007). We present here the solution as an expansion in modes, without giving details about finding the numerical solution [which can be found in Newman (1994); Sturova (2006)]. The single-frequency equations are based on assuming that all quantities are proportional to $e^{i\omega t}$, so that

$$\Phi(\mathbf{x},t) = \phi(\mathbf{x},\omega)\,e^{i\omega t}, \quad \Lambda_n(t) = \alpha_n(\omega)\,e^{i\omega t} \quad \text{and} \quad \xi(x,t) = \zeta(x,\omega)\,e^{i\omega t}, \tag{22}$$

where $\mathbf{x}=(x,z)$. Note that sometimes we will not write the dependence on ω explicitly. Under these assumptions, Eqs. (19a)–(19e) become

$$\Delta\phi = 0, \quad -h<z<0, \quad \partial_z\phi = 0, \quad z=-h, \tag{23a,b}$$

$$\omega^2\phi = \partial_z\phi, \quad x\notin(-b,b), \quad z=0, \tag{23c}$$

$$i\omega\sum_{n=0}^{\infty}\alpha_n w_n = \partial_z\phi, \quad x\in(-b,b), \quad z=0, \tag{23d}$$

$$\sum_{n=0}^{\infty}\alpha_n(1+\beta\lambda_n^4)w_n - \omega^2\gamma\sum_{n=0}^{\infty}\alpha_n w_n = -i\omega\phi, \quad x\in(-b,b), \quad z=0. \tag{23e}$$

We solve for the potential (and displacement) as the sum of the diffracted and radiation potentials in the standard way, as for a rigid body. We begin with the diffraction potential $\phi^{(d)}$ which satisfies the following equations:

$$\Delta\phi^{(d)} = 0, \quad -h<z<0, \tag{24a}$$

$$\partial_z\phi^{(d)} = 0, \quad z=-h, \tag{24b}$$

$$\partial_z\phi^{(d)} = \omega^2\phi^{(d)}, \quad x\notin(-b,b), \quad z=0, \tag{24c}$$

$$\partial_z\phi^{(d)} = 0, \quad x\in(-b,b), \quad z=0. \tag{24d}$$

Furthermore, $\phi^{(d)}$ satisfies the radiation condition

$$\frac{\partial}{\partial x}(\phi^{(d)}-\phi_\kappa^{\mathrm{In}}) \pm ik(\phi^{(d)}-\phi_\kappa^{\mathrm{In}}) = 0 \quad \text{as} \quad x\to\pm\infty, \tag{25}$$

where k is the wavenumber, which is the positive real solution of the dispersion equation

$$k\tanh(kh) = \omega^2, \tag{26}$$

and $\phi_\kappa^{\rm In}$ is the incident wave given by

$$\phi_\kappa^{\rm In} = \frac{{\rm i}\omega}{k \sinh kh} \cosh k(z+h)\, {\rm e}^{{\rm i}\kappa k x} \tag{27}$$

(which has unit amplitude in displacement) and κ is either 1 for a wave travelling towards negative infinity or -1 for a wave travelling towards positive infinity (we will need both these solutions). We now consider the radiation potentials $\phi^{(n)}$, which satisfy the following equations:

$$\Delta\phi^{(n)} = 0, \quad -h<z<0, \tag{28a}$$

$$\partial_z \phi^{(n)} = 0, \quad z=-h, \tag{28b}$$

$$\partial_z \phi^{(n)} = \omega^2 \phi^{(n)}, \quad x \notin (-b,b), \quad z=0 \tag{28c}$$

$$\partial_z \phi^{(n)} = {\rm i}\omega w_n, \quad x \in (-b,b), \quad z=0. \tag{28d}$$

The radiation condition for the radiation potential is

$$\frac{\partial \phi^{(n)}}{\partial x} \pm {\rm i}k\phi^{(n)} = 0 \quad \text{as } x \to \pm\infty. \tag{29}$$

The method used to solve these equations for deep water is described in Sturova (2006), and for finite depth in Newman (1994). Therefore we find the potential as

$$\phi = \phi_\kappa^{(d)} + \sum_{n=0}^{\infty} \alpha_{n,\kappa} \phi^{(n)}, \tag{30}$$

so that

$$\sum_{n=0}^{\infty} (1 + \beta\lambda_n^4 - \omega^2\gamma)\alpha_{n,\kappa} w_n = -{\rm i}\omega\phi_\kappa^{(d)} - {\rm i}\omega \sum_{n=0}^{\infty} \alpha_{n,\kappa}\phi^{(n)}. \tag{31}$$

If we multiply by w_m and take an inner product over the plate we obtain

$$(1 + \beta\lambda_n^4 - \omega^2\gamma)\alpha_{n,\kappa} = -{\rm i}\omega \int_{-b}^{b} \phi_\kappa^{(d)} w_n \, {\rm d}x + \sum_{m=0}^{\infty} (\omega^2 a_{mn}(\omega) - {\rm i}\omega b_{mn}(\omega))\alpha_{m,\kappa}, \tag{32}$$

where the real functions $a_{mn}(\omega)$ and $b_{mn}(\omega)$ are given by

$$\omega^2 a_{mn}(\omega) - {\rm i}\omega b_{mn}(\omega) = -{\rm i}\omega \int_{-b}^{b} \phi^{(m)} w_n \, {\rm d}x, \tag{33}$$

and they are referred to as the added mass and damping coefficients, respectively. Eq. (32) is solved by truncating the number of modes. The expression for the displacement is

$$\zeta_\kappa(x,\omega) = \begin{cases} \sum\limits_{n=0}^{\infty} \alpha_{n,\kappa}(\omega) w_n(x), & x \in (-b,b), \\ -{\rm i}\omega\left(\phi_\kappa^{(d)}(\mathbf{x},\omega)|_{z=0} + \sum\limits_{n=0}^{\infty} \alpha_{n,\kappa}\phi^{(n)}(\mathbf{x},\omega)|_{z=0} \right), & x \notin (-b,b). \end{cases} \tag{34}$$

4. Time domain solution by a spectral expansion

We briefly summarize the results of Hazard and Meylan (2007). The spectral solution consists of writing the equations of motion in the time domain (7a)–(7e) as

$$\partial_t^2 \xi + \partial_n \mathbf{H} \xi = 0, \tag{35}$$

where $\partial_n \mathbf{H}$ is the Dirichlet-to-Neumann map of ξ to $\partial_n \Psi$ at the surface, and Ψ satisfies

$$\Delta\Psi = 0, \quad -h<z<0, \tag{36a}$$

$$\partial_n \Psi = 0, \quad z=-h, \tag{36b}$$

$$\Psi = \xi, \quad x \notin (-b,b), \quad z=0, \tag{36c}$$

$$\Psi - \gamma \partial_n \Psi = \beta \partial_x^4 \xi + \xi, \quad x \in (-b,b), \quad z=0. \tag{36d}$$

We also require that ξ satisfies the free-edge conditions. The operator $\partial_n \mathbf{H}$ is positive and self-adjoint in the Hilbert space $\mathscr{H}$, with inner product

$$\langle \xi, \xi' \rangle_{\mathscr{H}} = \langle \xi, \xi' \rangle_{\mathbb{R}} + \beta \langle \partial_x^2 \xi, \partial_x^2 \xi' \rangle_P. \tag{37}$$

The eigenfunctions of $\partial_n \mathbf{H}$ are the single-frequency solutions $\zeta(x,\omega)$ given by (34), and the eigenvalue is ω^2. The key property is that they normalize as

$$\langle \zeta_\kappa(x,\omega), \zeta_{\kappa'}(x,\omega') \rangle_{\mathscr{H}} = 2\pi \frac{\mathrm{d}\omega}{\mathrm{d}k} \delta_{\kappa\kappa'} \delta(\omega - \omega'). \tag{38}$$

This property allows us to construct the spectral expansion of the solution, which is given by

$$\zeta(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^+} \sum_{\kappa=\pm 1} \cos(\omega t) \langle \xi_0, \zeta_\kappa \rangle_{\mathscr{H}} \zeta_\kappa \frac{\mathrm{d}k}{\mathrm{d}\omega} \, \mathrm{d}\omega. \tag{39}$$

Eq. (39) was the one used in Hazard and Meylan (2007). We want to obtain an expression in terms of the modes of the plate, and we therefore substitute the expression for the eigenfunctions (34) into (39) and consider the evolution of

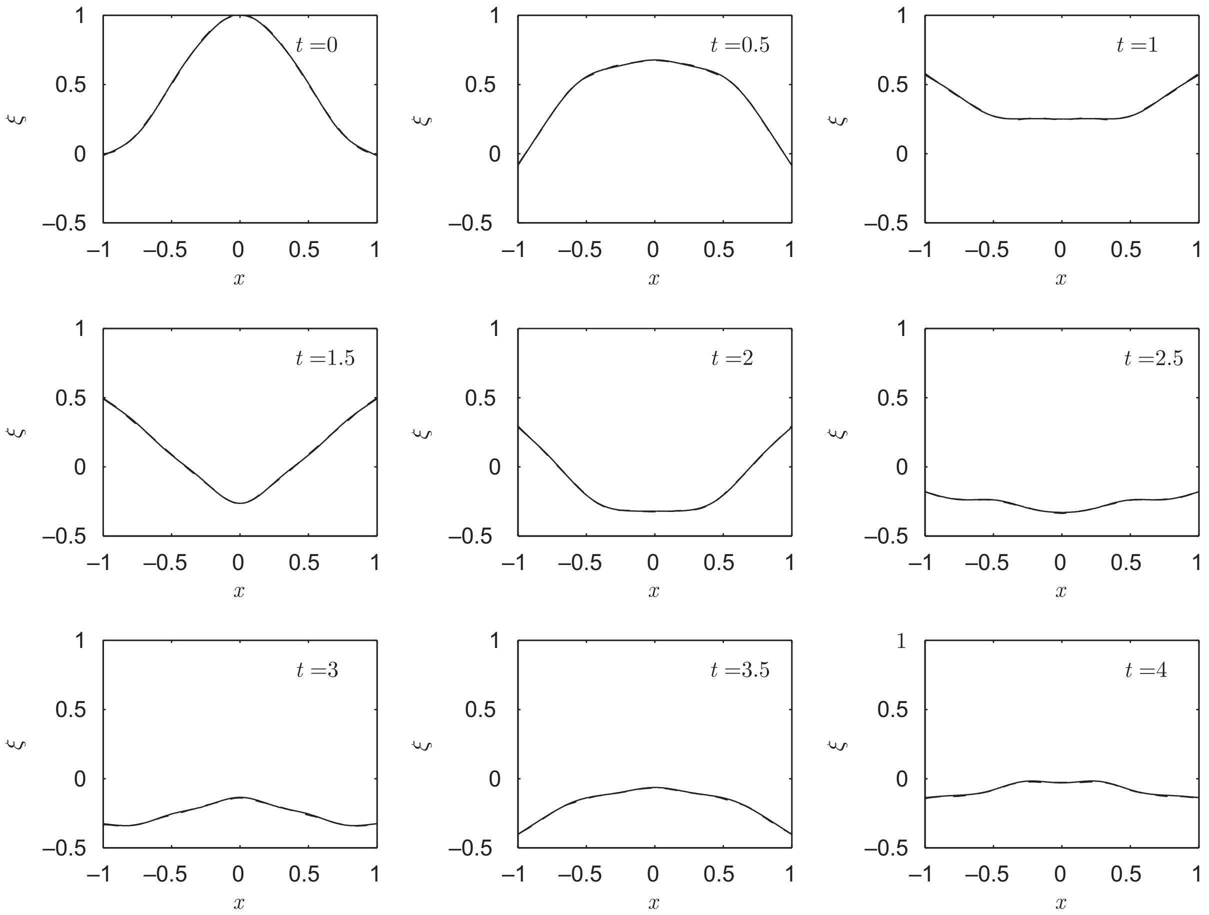

Fig. 1. Evolution of the plate with initial condition given by Eq. (63) with $\beta = \gamma = 0.005$ and $b = 1$. The water depth is infinite and $h = 16$. The solution for the solid curve is the finite depth theory and the dashed curve is the infinite water depth theory.

each mode separately, to obtain

$$\alpha_n(t) = \frac{1}{2\pi}\int_{\mathbb{R}^+} \sum_{\kappa=\pm 1} \cos(\omega t)\, \langle \xi_0, \zeta_\kappa \rangle_{\mathscr{H}} \alpha_{n,\kappa}(\omega) \frac{\mathrm{d}k}{\mathrm{d}\omega}\, \mathrm{d}\omega. \tag{40}$$

We substitute the expansion for the initial conditions (20) into the inner products, and obtain

$$\langle \xi_0, \zeta_\kappa \rangle_{\mathscr{H}} = \sum_{m=0}^{\infty} (1 + \beta\lambda_m^4) \Lambda_m(0) \alpha_m(\omega). \tag{41}$$

Therefore

$$\alpha_n(t) = \frac{1}{2\pi}\int_{\mathbb{R}^+} \sum_{\kappa=\pm 1} \cos(\omega t) \left(\sum_{m=0}^{\infty} (1 + \beta\lambda_m^4) \Lambda_m(0) \alpha_m(\omega) \right) \alpha_n(\omega) \frac{\mathrm{d}k}{\mathrm{d}\omega}\, \mathrm{d}\omega. \tag{42}$$

Eq. (42) is a generalized eigenfunction expansion for the plate motion, written in terms of the modes of vibration.

5. Fourier transform solution

We can solve the time-dependent equations by taking the one-sided Fourier transform (equivalent to the Laplace transform). This equation can then be transformed to give a derivation of the integral equation which was first given by

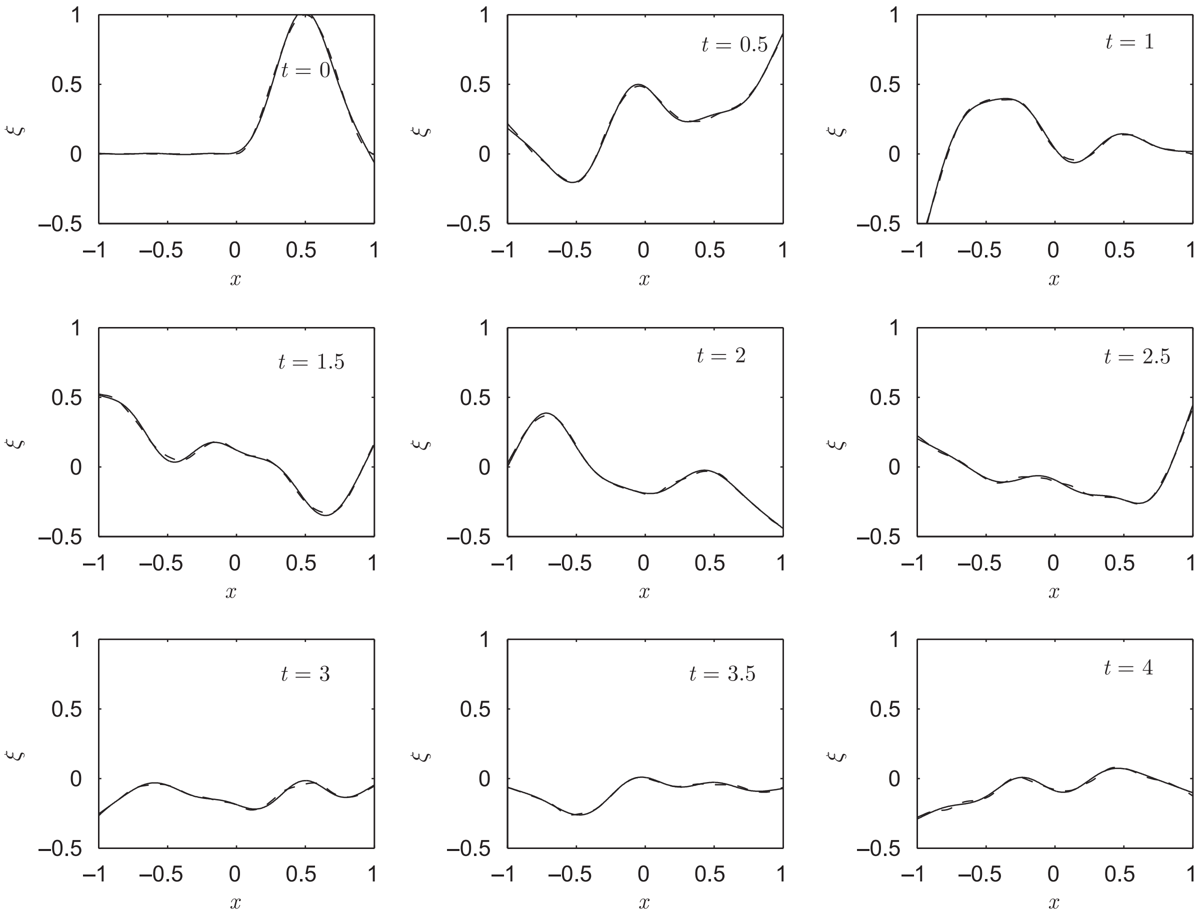

Fig. 2. As in Fig. 1, except that the initial condition is given by Eq. (64).

Ogilvie (1964) for a rigid body. If we take the one-sided Fourier transform, defined by

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{\mathrm{i}st} f(t)\,\mathrm{d}t \quad \text{and} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}st}\hat{f}(s)\,\mathrm{d}s \tag{43}$$

of Eqs. (19a)–(19e), we obtain

$$\Delta\hat{\Phi} = 0, \quad -h<z<0, \quad \partial_z\hat{\Phi} = 0, \quad z = -h, \tag{44a,b}$$

$$s^2\hat{\Phi} = \partial_z\hat{\Phi}, \quad x \notin (-b, b), \quad z = 0, \tag{44c}$$

$$\mathrm{i}s\sum_{n=0}^{\infty}\hat{\Lambda}_n w_n - \sum_{n=0}^{\infty}\Lambda_n(0)w_n = \partial_z\hat{\Phi}, \quad z = 0, \tag{44d}$$

$$\sum_{n=0}^{\infty}\hat{\Lambda}_n(1+\beta\lambda_n^4)w_n - s^2\gamma\sum_{n=0}^{\infty}\hat{\Lambda}_n w_n - \mathrm{i}s\gamma\sum_{n=0}^{\infty}\Lambda_n(0)w_n = -\mathrm{i}s\hat{\Phi}, \quad x \in (-b, b), \quad z = 0, \tag{44e}$$

where the hat denotes the Fourier/Laplace transform. We have assumed that the only initial condition is due to the plate displacement and the fluid is at rest.

Eqs. (44a)–(44d) can be solved using the solution for the radiation Eqs. (28a)–(29). This gives us

$$\hat{\Phi} = \sum_{n=0}^{\infty}(\hat{\Lambda}_n - \Lambda_n(0)/\mathrm{i}s)\phi^{(n)}, \tag{45}$$

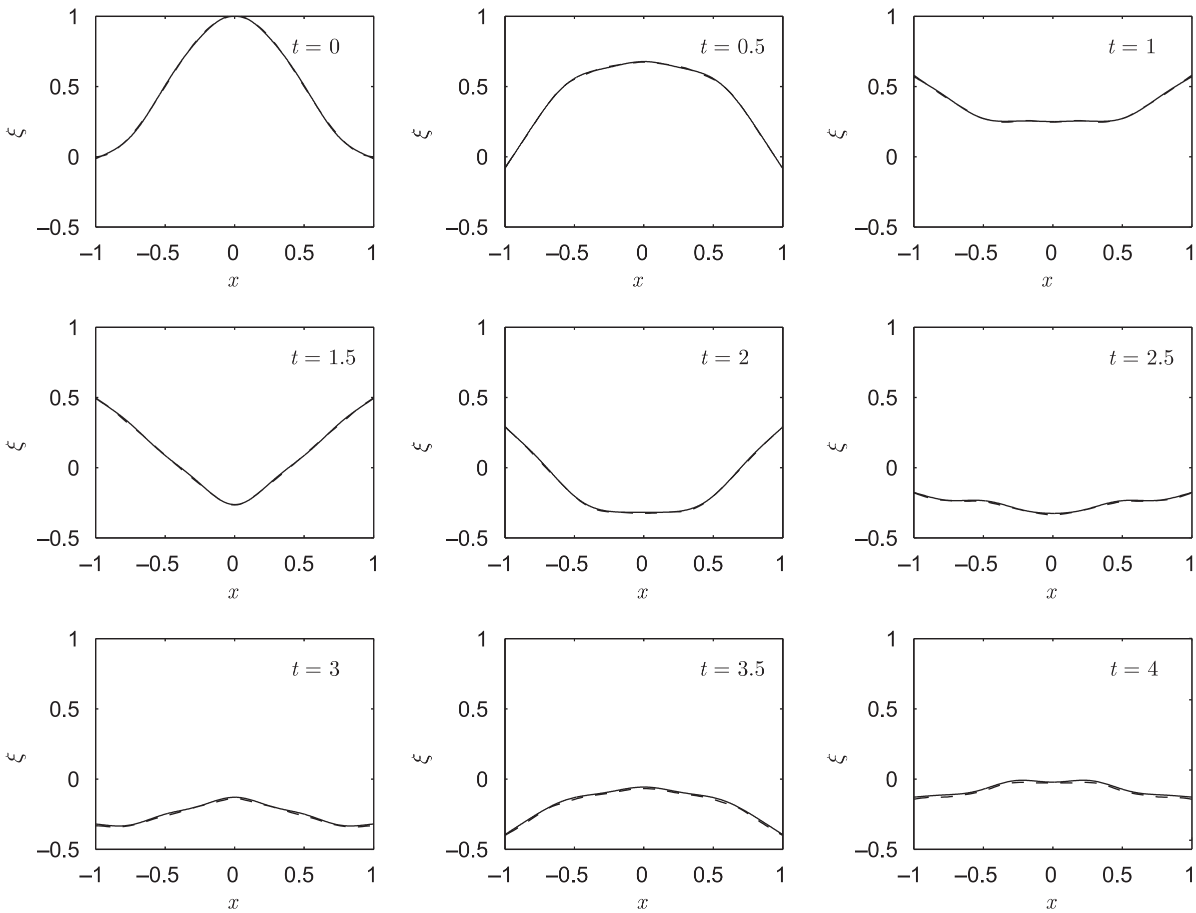

Fig. 3. As in Fig. 1, except that the finite water depth is $h = 8$.

and if we substitute this into Eq. (44e) and multiply by w_m and take an inner product over the plate the solution for $\hat{\Lambda}$ becomes

$$(1+\beta\lambda_n^4 - s^2\gamma)\hat{\Lambda}_n - \mathrm{i}s\gamma\Lambda_n(0) = \sum_{m=0}^{\infty}(s^2 a_{mn} - \mathrm{i}s b_{mn})(\hat{\Lambda}_m - \Lambda_m(0)/\mathrm{i}s). \tag{46}$$

The solution to Eq. (46) was calculated numerically in Meylan et al. (2004).

5.1. Memory effect equations of motion

We can write Eq. (46) as

$$(1+\beta\lambda_n^4)\hat{\Lambda}_n + \sum_{m=0}^{\infty}[(\gamma\delta_{mn} + a_{mn}(\infty))(-s^2\hat{\Lambda}_m - \mathrm{i}s\Lambda_m(0)) + ((a_{mn} - a_{mn}(\infty)) + b_{mn}/\mathrm{i}s)(-s^2\hat{\Lambda}_m - \mathrm{i}s\Lambda_m(0))] = 0, \tag{47}$$

where

$$a_{mn}(\infty) = \lim_{\omega\to\infty} a_{mn}(\omega), \tag{48}$$

and the reason for its inclusion will be shown shortly. We now introduce the function

$$L_{mn}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{b_{mn}(\omega)}{\omega}\sin(\omega t)\,\mathrm{d}\omega \tag{49}$$

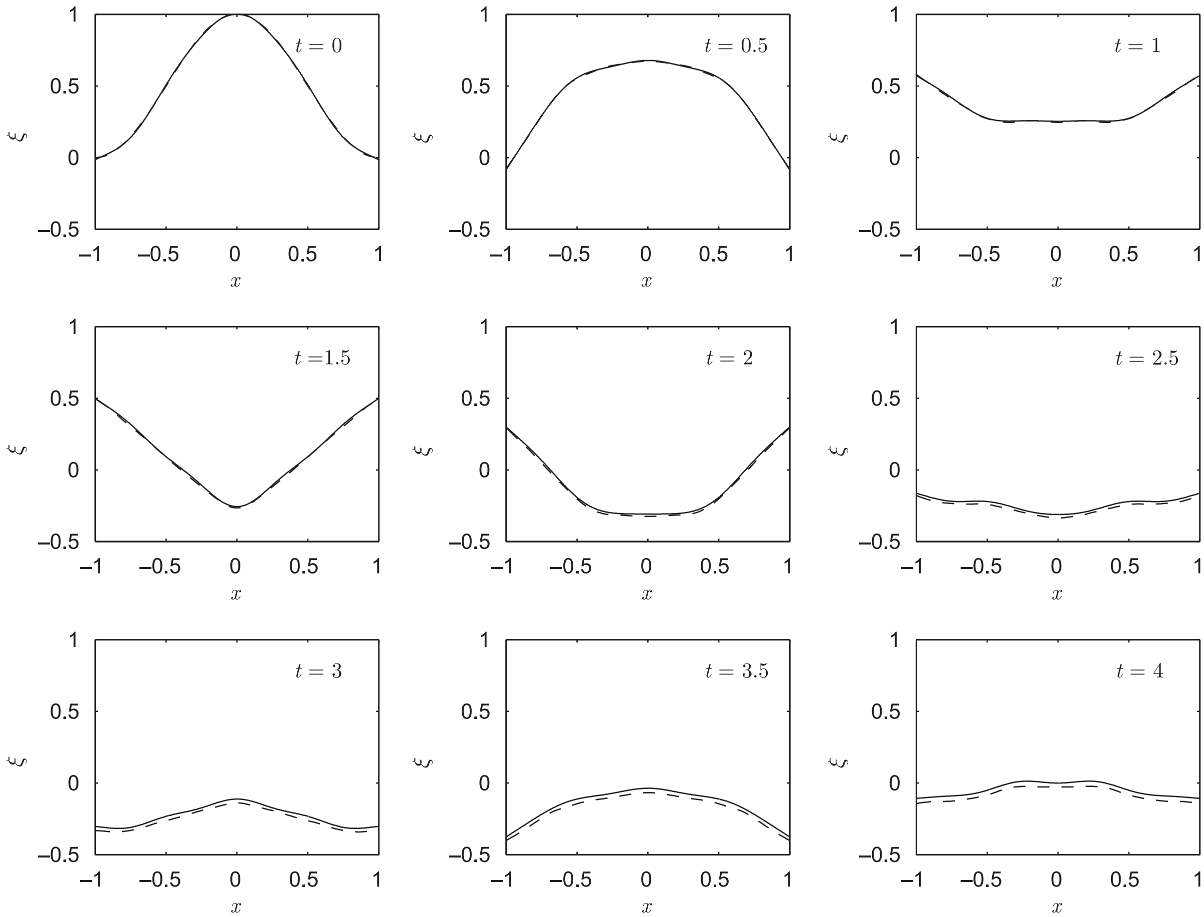

Fig. 4. As in Fig. 1, except that the finite water depth is $h = 4$.

whose Fourier transform is given by

$$\hat{L}_{mn} = (a_{mn}(s) - a_{mn}(\infty)) + b_{mn}(s)/\mathrm{i}s, \tag{50}$$

from Mei (1989, eq. (11.21)). Note that this result is connected with the Kramers–Krönig relations. Therefore

$$\sum_{m=0}^{\infty}[(\gamma\delta_{mn} + a_{mn}(\infty))(-s^2\hat{\Lambda}_m - \mathrm{i}s\Lambda_m(0)) + \hat{L}_{mn}(-s^2\hat{\Lambda}_m - \mathrm{i}s\Lambda_m(0))] + (1+\beta\lambda_n^4)\hat{\Lambda}_n = 0, \tag{51}$$

and if we take the inverse Fourier transform we obtain

$$\sum_{m=0}^{\infty}\left[(\gamma\delta_{mn} + a_{mn}(\infty))\partial_t^2\Lambda_m + \int_0^t \partial_\tau^2\Lambda_m(\tau)L_{mn}(t-\tau)\,\mathrm{d}\tau\right] + (1+\beta\lambda_n^4)\Lambda_n = 0. \tag{52}$$

This is the equation which was obtained by Cummins (1962) although the method to calculate L_{mn} was given later by Ogilvie (1964). Since we have assumed that the initial plate velocity is zero so that $\partial_t\Lambda_m(0) = 0$, we can use integration by parts to transform the equation to

$$\sum_{m=0}^{\infty}\left[(\gamma\delta_{mn} + a_{mn}(\infty))\partial_t^2\Lambda_m + \int_0^t \partial_\tau\Lambda_m(\tau)K_{mn}(t-\tau)\,\mathrm{d}\tau\right] + (1+\beta\lambda_n^4)\Lambda_n = 0, \tag{53}$$

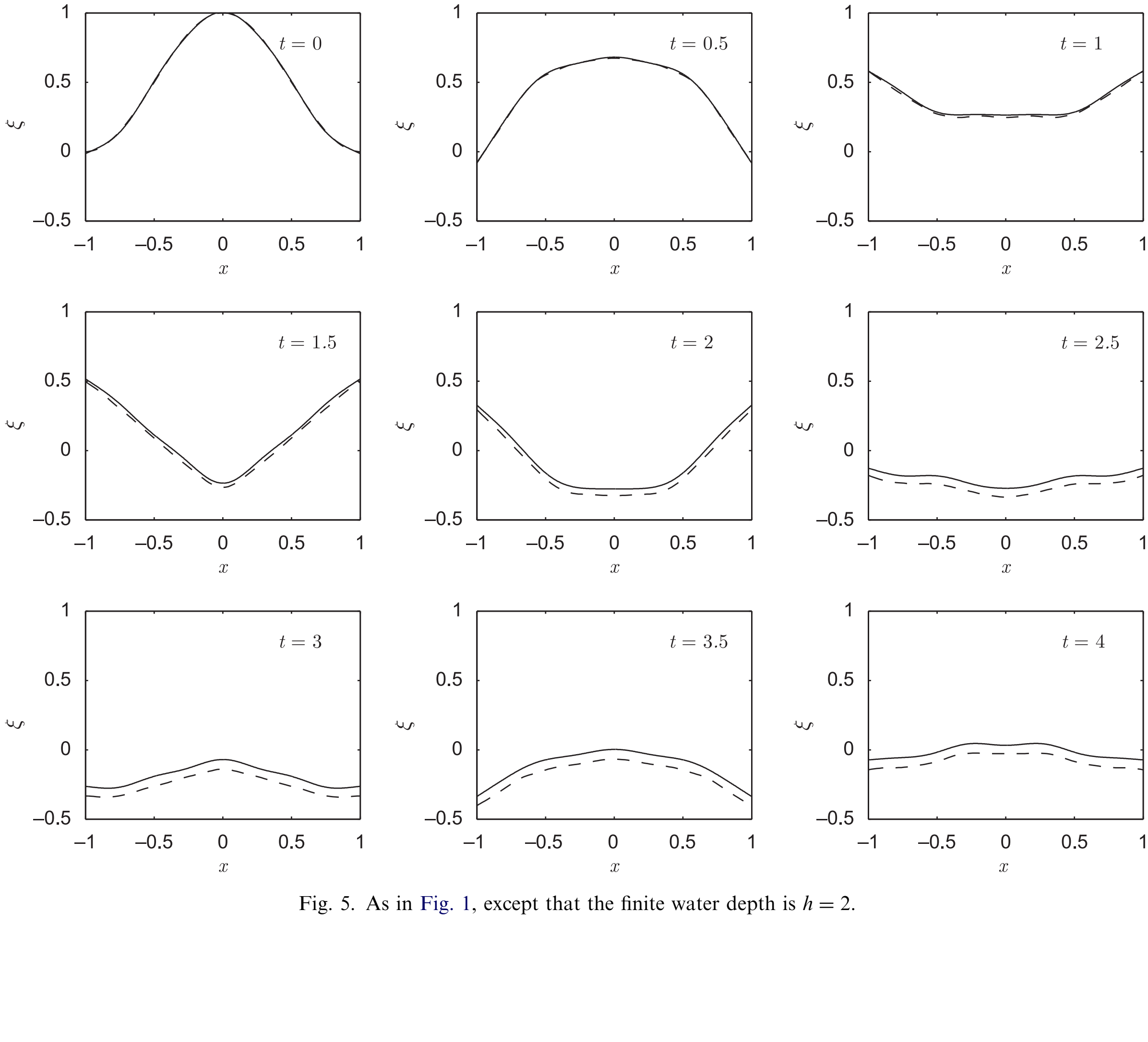

Fig. 5. As in Fig. 1, except that the finite water depth is $h = 2$.

where

$$K_{mn}(t) = \frac{2}{\pi}\int_0^\infty b_{mn}(\omega)\cos(\omega t)\,\mathrm{d}\omega. \tag{54}$$

Eq. (53) is the one used in Sturova (2006) and in the calculations presented in our results section.

Two expressions (42) and (53) solve the same problem, and we will see that they give the same solution numerically. However, while some similarities exist between the two expressions, no way to connect the two formulas is known to the authors, and finding such a connection remains an interesting open question.

6. Shallow water

The equations are considerably simplified under the assumption of shallow water. The solution method presented here is based on Sturova (2002), and an alternative approach based on the generalized eigenfunction expansion and also Lax–Philips scattering can be found in Meylan (2002). The equation takes the form

$$\partial_t \xi = -h\partial_x^2 \Phi, \tag{55a}$$

$$-\xi - \partial_t \Phi = 0, \quad x \notin (-b, b), \tag{55b}$$

$$-\xi - \partial_t \Phi = \beta\partial_x^4 \xi + \gamma\partial_t^2 \xi, \quad x \in (-b, b), \tag{55c}$$

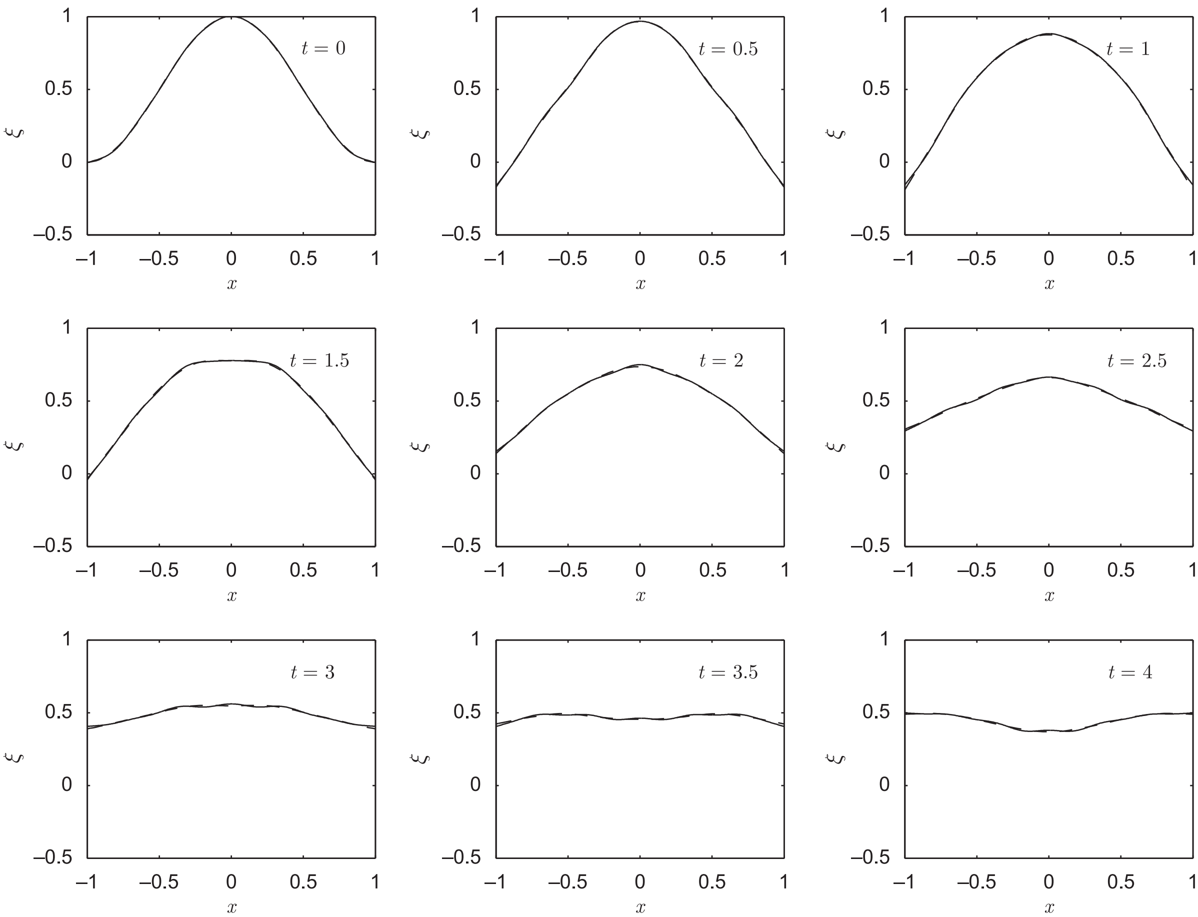

Fig. 6. As in Fig. 1, except that the water depth is $h = 0.02$ and the dashed curve is the shallow-depth solution.

where the potential is now only a function of x and t (see Sturova, 2002 for details). The free-edge boundary conditions are exactly as before (8), as are the initial conditions (9). If we substitute the expansion in modes (18) into Eq. (55c), multiply by $w_m(x)$, and integrate from $-b$ to b, we obtain the set of ordinary differential equations

$$\gamma\partial_t^2 \Lambda_m + [1+\beta\lambda_m^4]\Lambda_m + f_m(t) = 0, \tag{56}$$

where

$$f_m(t) = \int_{-b}^{b} \partial_t \Phi(x,t) w_m(x)\, dx. \tag{57}$$

A solution for $\Phi(x,t)$ is sought in the form

$$\Phi(x,t) = -\frac{1}{h}\left[\sum_{n=0}^{\infty} \partial_t \Lambda_n(t)\Psi_n(x) + q(x,t)\right], \tag{58}$$

where the functions $\Phi_n(x)$ satisfy the equation

$$\partial_x^2 \Psi_n(x) = w_n(x). \tag{59}$$

The function q is to be determined. According to Eq. (55a) the function q has the form $q(x,t) = xu(t) + v(t)$. The functions $u(t)$ and $v(t)$ are determined from the conditions of continuity of pressure and mass flow at $x = \pm b$.

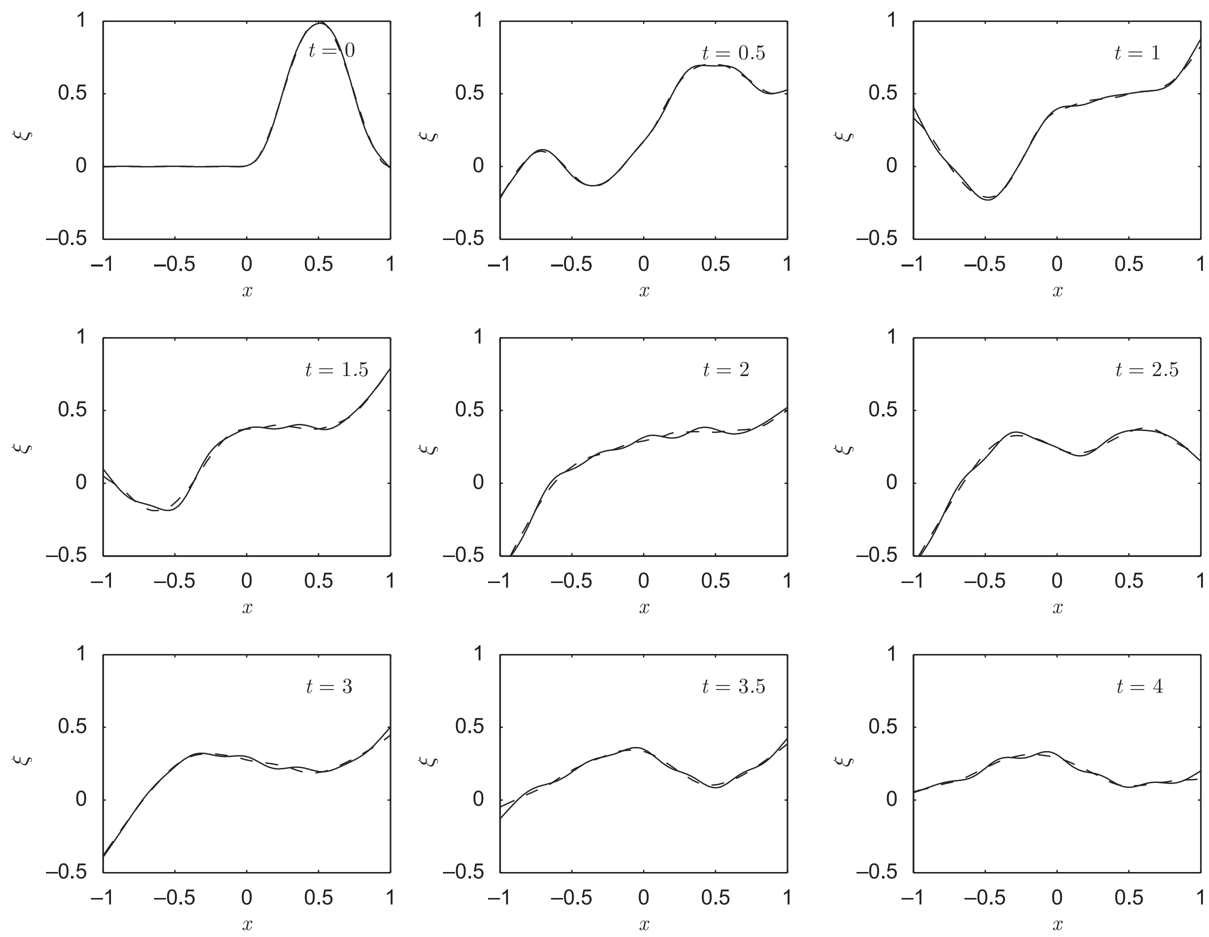

Fig. 7. As in Fig. 6, except that the initial condition is given by Eq. (64).

The final set of differential equations has the form

$$\sum_{n=0}^{\infty} \partial_t^2 \Lambda_n \left[\gamma \delta_{mn} - \frac{1}{h} C_{mn} + \frac{b^2}{2h} \left(\delta_{m0} + \frac{1}{3} \delta_{m1} \right) \right] + \frac{b}{\sqrt{h}} \left(\delta_{m0} \partial_t \Lambda_0 + \frac{1}{2} \delta_{m1} \partial_t \Lambda_1 \right) + \Lambda_m (1 + \beta \lambda_m^4) + \delta_{m1} \sqrt{\frac{2b}{3h}} u = 0, \tag{60a}$$

$$\partial_t u + \frac{1}{2\sqrt{2}} \left(\sqrt{\frac{b}{3}} \partial_t^2 \Lambda_1 + \sqrt{\frac{3h}{b}} \partial_t \Lambda_1 \right) + \frac{\sqrt{h}}{b} u = 0, \tag{60b}$$

where

$$C_{mn} = \int_{-b}^{b} w_m(x) \Psi_n(x) \, dx. \tag{61}$$

The analytical expressions for C_{mn} are given in Sturova (2002). This set of differential equations falls into two independent sets. The first set includes only the amplitude functions $\Lambda_{2k}(t)$ (even modes), whereas the second set includes the amplitude functions $\Lambda_{2k+1}(t)$ (odd modes) and the function $u(t)$.

We can write (60a) and (60b) in the form

$$\frac{d\vec{\Xi}}{dt} = \mathbf{M} \vec{\Xi}, \tag{62}$$

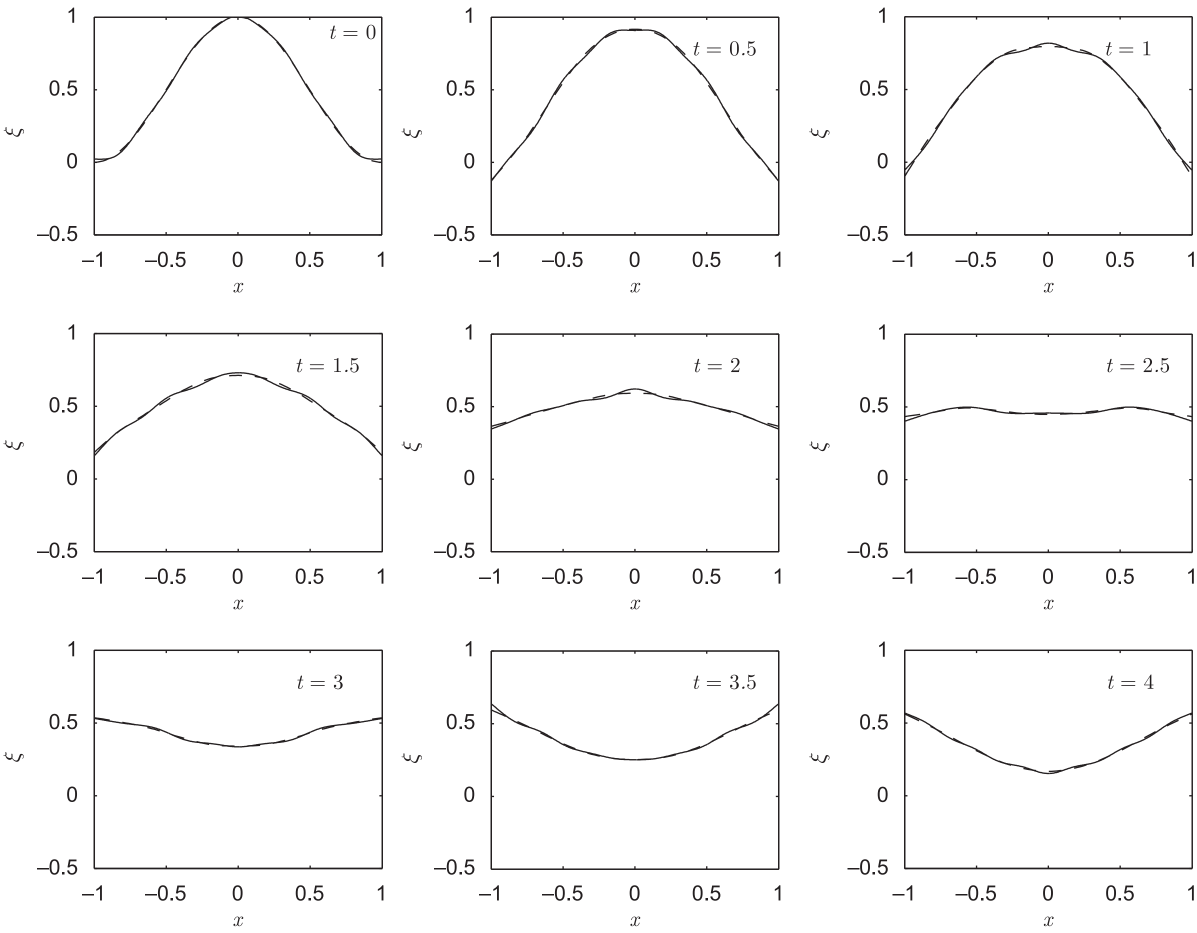

Fig. 8. As in Fig. 6, except that the depth is $h = 0.04$.

where **M** is a matrix and $\vec{\Xi} = (u, \Lambda_0, \Lambda_1, \ldots)$. We can therefore solve (60a) and (60b) by finding the eigenvalues and eigenvectors of **M**. An alternative method to find these eigenvalues using Lax–Philips scattering can be found in Meylan (2002), but this is not discussed here.

7. Numerical results

We present a series of benchmark calculations for the motion of an elastic floating plate released from rest, in the time-domain. By comparing the results from two different methods we can establish a high degree of confidence that the solution presented is correct. We also investigate the effect of depth of the motion, by comparing the solutions for shallow and infinite depth with the finite-depth solution.

We consider two initial displacements. The first is a symmetric displacement given by

$$\xi_0(x) = \tfrac{1}{2}(1 + \cos(\pi x/b)), \tag{63}$$

and the second is a nonsymmetric displacement given by

$$\xi_0(x) = \begin{cases} 0, & -b < x < 0, \\ \tfrac{1}{2}(1 + \cos(2\pi(x/b - 1/2))), & 0 < x < b. \end{cases} \tag{64}$$

In both cases the initial plate velocity is zero. We fix the plate parameters to be $\beta = \gamma = 0.005$ and $b = 1$, and we show the plate displacement for $t = 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4$. In all cases we use the first 20 modes. Fig. 1 shows the evolution of the plate for symmetric displacement (63) and Fig. 2 shows the displacement for the nonsymmetric initial

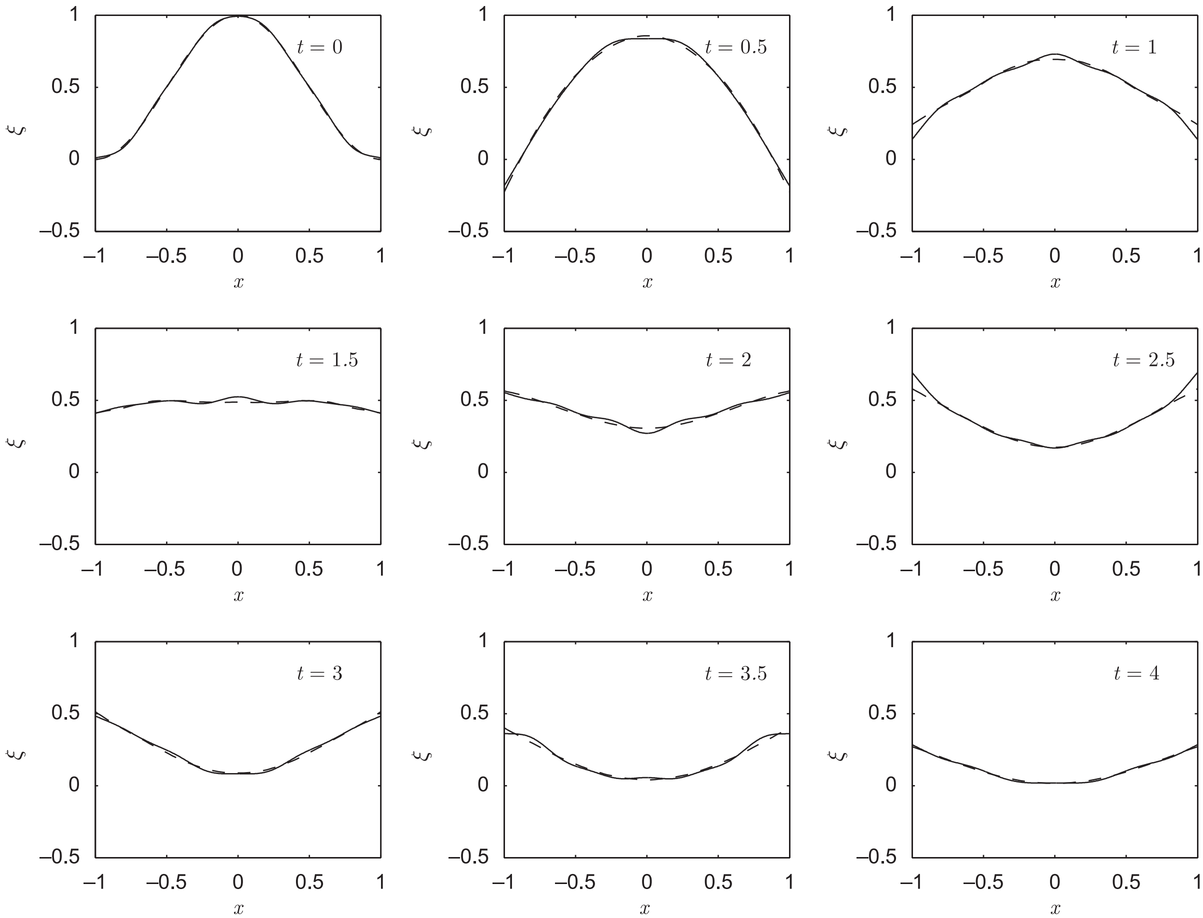

Fig. 9. As in Fig. 6, except that the depth is $h = 0.1$.

displacement (64). The finite depth solution is calculated with depth $h = 16$, which is sufficiently large that the finite and infinite depth solutions agree. The key result is that the two solution methods are identical. The complex response of the plate is apparent, and it is clear that the plate has almost come to rest by $t = 4$.

Figs. 3–5 show the same solution as Fig. 1, except the water depth is $h = 8, 4, 2$ respectively. We also plot the infinite-depth solution as a dashed line. The gradual change in the finite-depth solution from the infinite-depth solution is apparent in these figures, although even for $h = 2$ the infinite-depth solution is a good approximation.

We now consider comparison with the shallow-water solution. Fig. 6 shows the solution for depth $h = 0.02$, calculated by the finite-depth and shallow-depth theory for the symmetric initial displacement, and Fig. 7 shows the equivalent solution for the nonsymmetric initial displacement. The plate motion is significantly different from the infinite-depth solution shown in Figs. 1 and 2. It is also clear that the plate motion decays much more slowly in the case when the water depth is shallow.

Figs. 8 and 9 show the evolution of the plate displacement for the symmetric initial displacement for depths $h = 0.04$ and 0.1 (remember that the depth is still a parameter in the shallow-depth approximation). As the water depth increases we expect that the shallow water approximation should become less accurate, and this can be seen in the figures. However, the shallow water approximation is still working well for a depth of $h = 0.1$. It is also clear that, as the water depth is increased, the plate motion decays more rapidly.

8. Summary

We have presented three solution methods to calculate the motion of a floating two-dimensional plate, which is given some initial displacement and released from rest. The first solution method is based on the generalized eigenfunction expansion and it is valid for all depths. The second is based on the Fourier transformation and subsequent conversion of the equations to an integral equation. This is also valid for all depths, but the numerical solution is only calculated for infinite depth. The third method is a shallow-depth approximation. We presented numerical solutions which show good agreement, as expected in the appropriate depth regimes, therefore providing benchmark solutions. We also investigate the range of depths for which the approximations of infinite and finite water depth are valid.

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